FORMULAS AND THEOREMS FOR REFERENCE

I. Trigonometric Formulas

1. $\sin^2\theta + \cos^2\theta = 1$

2. $1 + \tan^2\theta = \sec^2\theta$

3. $1 + \cot^2\theta = \csc^2\theta$

4. $\sin(-\theta) = -\sin\theta$

5. $\cos(-\theta) = \cos\theta$

6. $\tan(-\theta) = -\tan\theta$

7. $\sin(A+B) = \sin A\cos B + \sin B\cos A$

8. $\sin(A-B) = \sin A\cos B - \sin B\cos A$

9. $\cos(A+B) = \cos A\cos B - \sin A\sin B$

10. $\cos(A-B) = \cos A\cos B + \sin A\sin B$

11. $\sin 2\theta = 2\sin\theta\cos\theta$

12. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

13. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\cot\theta}$

14. $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$

15. $\sec\theta = \dfrac{1}{\cos\theta}$

16. $\csc\theta = \dfrac{1}{\sin\theta}$

17. $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$

18. $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

II. Differentiation Formulas

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(fg) = fg' + gf'$

3. $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$

4. $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

5. $\frac{d}{dx}(\sin x) = \cos x$

6. $\frac{d}{dx}(\cos x) = -\sin x$

7. $\frac{d}{dx}(\tan x) = \sec^2 x$

8. $\frac{d}{dx}(\cot x) = -\csc^2 x$

9. $\frac{d}{dx}(\sec x) = \sec x \tan x$

10. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

11. $\frac{d}{dx}(e^x) = e^x \cdot x'$

12. $\frac{d}{dx}(a^x) = a^x \ln a \cdot x'$

13. $\frac{d}{dx}(\ln x) = \frac{1}{x} \cdot x'$

14. $\frac{d}{dx}(\text{ Arcsin } x) = \frac{1}{\sqrt{1-x^2}}$

15. $\frac{d}{dx}(\text{ Arctan } x) = \frac{1}{1+x^2}$

16. $\frac{d}{dx}(\text{Arcsec}\, x) = \frac{1}{|x|\sqrt{x^2-1}}$

III. Integration Formulas

1. $\int a\,dx = ax + C$

2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \dfrac{1}{x}\,dx = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C$

6. $\int \ln x\,dx = x\ln x - x + C$

7. $\int \sin x\,dx = -\cos x + C$

8. $\int \cos x\,dx = \sin x + C$

9. $\int \tan x\,dx = \ln|\sec x| + C$ or $-\ln|\cos x| + C$

10. $\int \cot x\,dx = \ln|\sin x| + C$

11. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

12. $\int \csc x\,dx = \ln|\csc x - \cot x| + C$

13. $\int \sec^2 x\,dx = \tan x + C$

14. $\int \sec x \tan x\,dx = \sec x + C$

15. $\int \csc^2 x\,dx = -\cot x + C$

16. $\int \csc x \cot x\,dx = -\csc x + C$

17. $\int \tan^2 x\,dx = \tan x - x + C$

18. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\overset{\tan^{-1}}{\text{Arctan}}\left(\dfrac{x}{a}\right) + C$

19. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \text{Arcsin}\left(\dfrac{x}{a}\right) + C$

IV. Formulas and Theorems

1. A function $y = f(x)$ is <u>continuous</u> at $x = a$ if i) $f(a)$ exists, ii) $\lim_{x \to a} f(x)$ exists, and iii) $\lim_{x \to a} = f(a)$.

2. <u>Even and Odd Functions</u>

   1. A function $y = f(x)$ is <u>even</u> if $f(-x) = f(x)$ for every x in the function's domain. Every even function is symmetric about the y-axis.
   2. A function $y = f(x)$ is <u>odd</u> if $f(-x) = -f(x)$ for every x in the function's domain. Every odd function is symmetric about the origin.

3. <u>Periodicity</u>

   A function $f(x)$ is periodic with period p $(p > 0)$ if $f(x+p) = f(x)$ for every value of x.

   <u>Note:</u> The period of the function $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ is $\frac{2\pi}{|B|}$. The amplitude is $|A|$. (The period of $y = \tan x$ is π).

4. <u>Intermediate-Value Theorem</u>

   A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

   <u>Note:</u> If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a, b).

5. <u>Limits of Rational Functions as $x \to \pm\infty$</u>

   i) $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

   <u>Example:</u> $\lim_{x \to \infty} \frac{x^2 - 2x}{x^3 + 3} = 0$

   ii) $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is infinite if the degree of $f(x) >$ the degree of $g(x)$

   <u>Example:</u> $\lim_{x \to +\infty} \frac{x^3 + 2x}{x^2 - 8} = \infty$

   iii) $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

   <u>Note:</u> The limit will be the ratio of the leading coefficient of $f(x)$ to $g(x)$.

   <u>Example:</u> $\lim_{x \to \infty} \frac{2x^2 - 3x + 2}{10x - 5x^2} = -\frac{2}{5}$

6. <u>Horizontal and Vertical Asymptotes</u>

1. A line $y = b$ is a <u>horizontal asymptote</u> of the graph of $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b.$$

2. A line $x = a$ is a <u>vertical asymptote</u> of the graph of $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} = \pm\infty.$$

7. <u>Average and Instantaneous Rate of Change</u>

i) <u>Average Rate of Change:</u> If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \boxed{\frac{\Delta y}{\Delta x}}$$

ii) <u>Instantaneous Rate of Change:</u> If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

8. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

9. <u>The Number e as a limit</u>

i) $$\lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n = e$$

ii) $$\lim_{n\to 0} (1+n)^{\frac{1}{n}} = e$$

10. <u>Rolle's Theorem</u>

If f is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a, b) such that $f'(c) = 0$.

11. <u>Mean Value Theorem</u>

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

12. <u>Extreme-Value Theorem</u>

If f is continuous on a closed interval $[a, b]$, then $f(x)$ has both a maximim and a minimum on $[a, b]$.

13. To find the maximum and minimum values of a function $y = f(x)$, locate

1. the points where $f'(x)$ is zero <u>or</u> where $f'(x)$ fails to exist
2. the end points, if any, on the domain of $f(x)$.

<u>Note:</u> These are the <u>only</u> candidates for the value of x where $f(x)$ may have a maximum or a minimum.

14. Let f be differentiable for $a < x < b$ and continuous for $a \le x \le b$.

1. If $f'(x) > 0$ for every x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for every x in (a, b), then f is decreasing on $[a, b]$.

15. Suppose that $f''(x)$ exists on the interval (a, b).

1. If $f''(x) > 0$ in (a, b), then f is concave upward in (a, b).
2. If $f''(x) < 0$ in (a, b), then f is concave downward in (a, b).

To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ <u>or</u> where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

16a. If a function is differentiable at a point $x = a$, it is continuous at that point. The converse is false, i.e. continuity does <u>not</u> imply differentiability.

16b. <u>Linear Approximation</u>

The linear approximation to $f(x)$ near $x = x_0$ is given by $y = f(x_0) + f'(x_0)(x - x_0)$ for x sufficiently close to x_0.

17. <u>Newton's Method</u>

Let f be a differentiable function and suppose r is a real zero of f. If x_n is an approximation to r, then the next approximation x_{n+1} is given by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n) \neq 0$. Successive approximations can be found using this method.

18. <u>L'Hôpital's Rule</u>

If $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, and if $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$.

19. Inverse Functions

1. If f and g are two functions such that $f(g(x)) = x$ for every x in the domain of g, and, $g(f(x)) = x$, for every x in the domain of f, then, f and g are inverse functions of each other.

2. A function f has an inverse if and only if no horizontal line intersects its graph more than once.

3. If f is either increasing or decreasing in an interval, then f has an inverse.

4. If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(\mathrm{I})$ and $g'(f(x)) = \dfrac{1}{f'(x)}$.

20. Properties of $y = e^x$

1. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.

2. The domain is the set of all real numbers, $-\infty < x < \infty$.

3. The range is the set of all positive numbers, $y > 0$.

4. $\dfrac{d}{dx}(e^x) = e^x$.

5. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$.

6. $y = e^x$ is continuous, increasing, and concave up for all x.

7. $\lim\limits_{x \to +\infty} e^x = +\infty$ and $\lim\limits_{x \to -\infty} e^x = 0$.

8. $e^{\ln x} = x$, for $x > 0$; $\ln(e^x) = x$ for all x.

21. Properties of $y = \ln x$

1. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.

2. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.

3. $y = \ln x$ is continuous and increasing everywhere on its domain.

4. $\ln(ab) = \ln a + \ln b$.

5. $\ln(a/b) = \ln a - \ln b$.

6. $\ln a^r = r \ln a$.

7. $y = \ln x < 0$ if $0 < x < 1$.

8. $\lim\limits_{x \to +\infty} \ln x = +\infty$ and $\lim\limits_{x \to 0^+} \ln x = -\infty$.

9. $\log_a x = \dfrac{\ln x}{\ln a}$.

22. Trapezoidal Rule

If a function f is continuous on the closed interval $[a,b]$ where $[a,b]$ has been partitioned into n subintervals $[x_0,x_1]$, $[x_1,x_2]$, ..., $[x_{n-1},x_n]$, each of length $(b-a)/n$,

then $\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)]$

23. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

i) $\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c

ii) $\int_a^a f(x)\,dx = 0$

iii) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

iv) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where f is continuous on an interval containing the numbers a, b, and c

v) If $f(x)$ is an odd function, then $\int_{-a}^a f(x)\,dx = 0$

vi) If $f(x)$ is an even function, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

vii) If $f(x)\geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$

viii) If $g(x)\geq f(x)$ on $[a,b]$, then $\int_a^b g(x)\,dx \geq \int_a^b f(x)\,dx$

24. Fundamental Theorem of Calculus:

$\int_a^b f(x)\,dx = F(b)-F(a)$, where $F'(x)=f(x)$, or $\frac{d}{dx}\int_a^x f(x)\,dx = f(x)$.

25. 1. If a particle moving along a straight line has a positive function $x(t)$, then its instantaneous velocity $v(t)=x'(t)$ and its acceleration $a(t)=v'(t)$.

2. $v(t)=\int a(t)\,dt$ and $x(t)=\int v(t)\,dt$.

26. The average value of $f(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

27. If f and g are continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves is $\int_a^b [f(x) - g(x)]\, dx$.

28. <u>Integration By "Parts"</u>

If $u = f(x)$ and $v = g(x)$ and if $f'(x)$ and $g'(x)$ are continuous, then $\int u\, dv = uv - \int v\, du$.

<u>Note:</u> The goal of the procedure is to choose u and dv so that $\int v\, du$ is easier to solve than the original problem.

<u>Suggestion:</u>

When "choosing" u, remember L. I. A.T.E., where L is the logarithmic function, I is an inverse trigonometric function, A is an algebraic function, T is a trigonometric function, and E is the exponential function. Just choose u as the first expression in L. I. A.T.E. (and dv will be the remaining part of the integrand). For example, when integrating $\int x \ln x\, dx$, choose $u = \ln x$, since L comes first in L. I. A.T.E., and $dv = x dx$. When integrating $\int xe^x\, dx$, choose $u = x$, since x is an algebraic function, and A comes before E in L. I. A.T.E., and $dv = e^x\, dx$. One more example, when integrating $\int x \operatorname{Arctan} x\, dx$, let $u = \operatorname{Arctan} x$, since I comes before A in L. I. A.T.E., and $dv = x\, dx$.

29a. <u>Volumes of Solids of Revolution</u>

Let f be nonnegative and continuous on $[a, b]$, and let R be the region bounded above by $y = f(x)$, below by the x-axis, and the sides by the lines $x = a$ and $x = b$.

1. When this region R is revolved about the x-axis, it generates a solid (having circular cross sections) whose volume $V = \int_a^b \pi [f(x)]^2\, dx$.

2. When R is revolved about the y-axis, it generates a solid whose volume $V = \int_a^b 2\pi x f(x)\, dx$.

29b. <u>Volumes of Solids with Known Cross Sections</u>

1. For cross sections of area $A(x)$, taken perpendicular to the x-axis, volume $= \int_a^b A(x)\, dx$.

2. For cross sections of area $A(y)$, taken perpendicular to the y-axis, volume $= \int_c^d A(y)\, dy$.

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7. $\sin(A+B) = \sin A\cos B + \sin B\cos A$

8. $\sin(A-B) = \sin A\cos B - \sin B\cos A$

9. $\cos(A+B) = \cos A\cos B - \sin A\sin B$

10. $\cos(A-B) = \cos A\cos B + \sin A\sin B$

11. $\sin 2\theta = 2\sin\theta\cos\theta$

12. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

13. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\cot\theta}$

14. $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$

15. $\sec\theta = \dfrac{1}{\cos\theta}$

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4. $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

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8. $\frac{d}{dx}(\cot x) = -\csc^2 x$

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11. $\frac{d}{dx}(e^x) = e^x \cdot x'$

12. $\frac{d}{dx}(a^x) = a^x \ln a \cdot x'$

13. $\frac{d}{dx}(\ln x) = \frac{1}{x} \cdot x'$

14. $\frac{d}{dx}(\text{Arcsin } x) = \frac{1}{\sqrt{1-x^2}}$

15. $\frac{d}{dx}(\text{Arctan } x) = \frac{1}{1+x^2}$

16. $\frac{d}{dx}(\text{Arcsec } x) = \frac{1}{|x|\sqrt{x^2-1}}$

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1. $\int a\,dx = ax + C$

2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1$

3. $\int \dfrac{1}{x}\,dx = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C$

6. $\int \ln x\,dx = x\ln x - x + C$

7. $\int \sin x\,dx = -\cos x + C$

8. $\int \cos x\,dx = \sin x + C$

9. $\int \tan x\,dx = \ln|\sec x| + C$ or $-\ln|\cos x| + C$

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11. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

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13. $\int \sec^2 x\,dx = \tan x + C$

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18. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\overset{\tan^{-1}}{\text{Arctan}}\left(\dfrac{x}{a}\right) + C$

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1. A function $y = f(x)$ is continuous at $x = a$ if i) $f(a)$ exists, ii) $\lim_{x\to a} f(x)$ exists, and iii) $\lim_{x\to a} = f(a)$.

2. Even and Odd Functions

   1. A function $y = f(x)$ is even if $f(-x) = f(x)$ for every x in the function's domain. Every even function is symmetric about the y–axis.
   2. A function $y = f(x)$ is odd if $f(-x) = -f(x)$ for every x in the function's domain. Every odd function is symmetric about the origin.

3. Periodicity

   A function $f(x)$ is periodic with period p $(p > 0)$ if $f(x+p) = f(x)$ for every value of x.

   Note: The period of the function $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ is $\dfrac{2\pi}{|B|}$.
   The amplitude is $|A|$. (The period of $y = \tan x$ is π).

4. Intermediate-Value Theorem

   A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

   Note: If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a, b).

5. Limits of Rational Functions as $x \to \pm\infty$

   i) $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

   Example: $\lim_{x\to\infty} \dfrac{x^2 - 2x}{x^3 + 3} = 0$

   ii) $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)}$ is infinite if the degree of $f(x) >$ the degree of $g(x)$

   Example: $\lim_{x\to+\infty} \dfrac{x^3 + 2x}{x^2 - 8} = \infty$

   iii) $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

   Note: The limit will be the ratio of the leading coefficient of $f(x)$ to $g(x)$.

   Example: $\lim_{x\to\infty} \dfrac{2x^2 - 3x + 2}{10x - 5x^2} = -\dfrac{2}{5}$

6. Horizontal and Vertical Asymptotes

1. A line $y = b$ is a horizontal asymptote of the graph of $y = f(x)$ if either

$\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

2. A line $x = a$ is a vertical asymptote of the graph of $y = f(x)$ if either

$\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} = \pm\infty$.

7. Average and Instantaneous Rate of Change

i) Average Rate of Change: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \boxed{\frac{\Delta y}{\Delta x}}.$$

ii) Instantaneous Rate of Change: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

8. $$f'(x) = \lim_{h\to0} \frac{f(x+h) - f(x)}{h}$$

9. The Number e as a limit

i) $\lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n = e$

ii) $\lim_{n\to0} (1+n)^{\frac{1}{n}} = e$

10. Rolle's Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a, b) such that $f'(c) = 0$.

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If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

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If f is continuous on a closed interval $[a, b]$, then $f(x)$ has both a maximim and a minimum on $[a, b]$.

13. To find the maximum and minimum values of a function $y = f(x)$, locate

1. the points where $f'(x)$ is zero <u>or</u> where $f'(x)$ fails to exist
2. the end points, if any, on the domain of $f(x)$.

<u>Note:</u> These are the <u>only</u> candidates for the value of x where $f(x)$ may have a maximum or a minimum.

14. Let f be differentiable for $a < x < b$ and continuous for $a \leq x \leq b$.

1. If $f'(x) > 0$ for every x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for every x in (a, b), then f is decreasing on $[a, b]$.

15. Suppose that $f''(x)$ exists on the interval (a, b).

1. If $f''(x) > 0$ in (a, b), then f is concave upward in (a, b).
2. If $f''(x) < 0$ in (a, b), then f is concave downward in (a, b).

To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ <u>or</u> where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

16a. If a function is differentiable at a point $x = a$, it is continuous at that point. The converse is false, i.e. continuity does <u>not</u> imply differentiability.

16b. <u>Linear Approximation</u>

The linear approximation to $f(x)$ near $x = x_0$ is given by $y = f(x_0) + f'(x_0)(x - x_0)$ for x sufficiently close to x_0.

17. <u>Newton's Method</u>

Let f be a differentiable function and suppose r is a real zero of f. If x_n is an approximation to r, then the next approximation x_{n+1} is given by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n) \neq 0$. Successive approximations can be found using this method.

18. <u>L'Hôpital's Rule</u>

If $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, and if $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$.

19. Inverse Functions

1. If f and g are two functions such that $f(g(x)) = x$ for every x in the domain of g, and, $g(f(x)) = x$, for every x in the domain of f, then, f and g are inverse functions of each other.
2. A function f has an inverse if and only if no horizontal line intersects its graph more than once.
3. If f is either increasing or decreasing in an interval, then f has an inverse.
4. If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(\mathrm{I})$ and $g'(f(x)) = \dfrac{1}{f'(x)}$.

20. Properties of $y = e^x$

1. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.
2. The domain is the set of all real numbers, $-\infty < x < \infty$.
3. The range is the set of all positive numbers, $y > 0$.
4. $\dfrac{d}{dx}(e^x) = e^x$.
5. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$.
6. $y = e^x$ is continuous, increasing, and concave up for all x.
7. $\lim\limits_{x\to+\infty} e^x = +\infty$ and $\lim\limits_{x\to-\infty} e^x = 0$.
8. $e^{\ln x} = x$, for $x > 0$; $\ln(e^x) = x$ for all x.

21. Properties of $y = \ln x$

1. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.
2. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.
3. $y = \ln x$ is continuous and increasing everywhere on its domain.
4. $\ln(ab) = \ln a + \ln b$.
5. $\ln(a/b) = \ln a - \ln b$.
6. $\ln a^r = r \ln a$.
7. $y = \ln x < 0$ if $0 < x < 1$.
8. $\lim\limits_{x\to+\infty} \ln x = +\infty$ and $\lim\limits_{x\to0^+} \ln x = -\infty$.
9. $\log_a x = \dfrac{\ln x}{\ln a}$.

22. Trapezoidal Rule

If a function f is continuous on the closed interval $[a,b]$ where $[a,b]$ has been partitioned into n subintervals $[x_0,x_1]$, $[x_1,x_2]$, $\ldots$, $[x_{n-1},x_n]$, each of length $(b-a)/n$,

then $\displaystyle\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)]$

23. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

i) $\displaystyle\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c

ii) $\displaystyle\int_a^a f(x)\,dx = 0$

iii) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

iv) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where f is continuous on an interval containing the numbers a, b, and c

v) If $f(x)$ is an odd function, then $\displaystyle\int_{-a}^a f(x)\,dx = 0$

vi) If $f(x)$ is an even function, then $\displaystyle\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

vii) If $f(x)\geq 0$ on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$

viii) If $g(x)\geq f(x)$ on $[a,b]$, then $\displaystyle\int_a^b g(x)\,dx \geq \int_a^b f(x)\,dx$

24. Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b)-F(a), \text{ where } F'(x)=f(x), \text{ or } \frac{d}{dx}\int_a^x f(x)\,dx = f(x).$$

25. 1. If a particle moving along a straight line has a positive function $x(t)$, then its instantaneous velocity $v(t)=x'(t)$ and its acceleration $a(t)=v'(t)$.

2. $v(t)=\displaystyle\int a(t)\,dt$ and $x(t)=\displaystyle\int v(t)\,dt$.

26. The average value of $f(x)$ on $[a,b]$ is $\displaystyle\frac{1}{b-a}\int_a^b f(x)\,dx$.

27. If f and g are continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves is $\int_a^b [f(x) - g(x)]\, dx$.

28. Integration By "Parts"

If $u = f(x)$ and $v = g(x)$ and if $f'(x)$ and $g'(x)$ are continuous, then $\int u\, dv = uv - \int v\, du$.

Note: The goal of the procedure is to choose u and dv so that $\int v\, du$ is easier to solve than the original problem.

Suggestion:

When "choosing" u, remember L. I. A.T.E., where L is the logarithmic function, I is an inverse trigonometric function, A is an algebraic function, T is a trigonometric function, and E is the exponential function. Just choose u as the first expression in L. I. A.T.E. (and dv will be the remaining part of the integrand). For example, when integrating $\int x \ln x\, dx$, choose $u = \ln x$, since L comes first in L. I. A.T.E., and $dv = x dx$. When integrating $\int xe^x\, dx$, choose $u = x$, since x is an algebraic function, and A comes before E in L. I. A.T.E., and $dv = e^x\, dx$. One more example, when integrating $\int x \operatorname{Arctan} x\, dx$, let $u = \operatorname{Arctan} x$, since I comes before A in L. I. A.T.E., and $dv = x\, dx$.

29a. Volumes of Solids of Revolution

Let f be nonnegative and continuous on $[a, b]$, and let R be the region bounded above by $y = f(x)$, below by the x-axis, and the sides by the lines $x = a$ and $x = b$.

1. When this region R is revolved about the x-axis, it generates a solid (having circular cross sections) whose volume $V = \int_a^b \pi [f(x)]^2\, dx$.
2. When R is revolved about the y-axis, it generates a solid whose volume $V = \int_a^b 2\pi x f(x)\, dx$.

29b. Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$, taken perpendicular to the x-axis, volume $= \int_a^b A(x)\, dx$.
2. For cross sections of area $A(y)$, taken perpendicular to the y-axis, volume $= \int_c^d A(y)\, dy$.